Dynamic Models of Investment

Convex Adjustment Costs and the Determinants of Investment under Certainty and Uncertainty
The Optimal Capital Stock

• So far we have been assuming that firms choose their capital stock so that the marginal product of capital equals the user cost of capital, as determined by the real interest rate and the rate of depreciation.

• This theory in fact determines the optimal capital stock and not the amount of investment.

• Investment specifies how quickly a firm moves from its current to the optimal capital stock. When firms can adjust their capital stock immediately and without cost, the flow of investment is not defined, as the capital stock jumps immediately to its optimal level.
Adjustment Costs and the Investment Function

• In fact, however, the change of the capital stock involves adjustment costs.

• A firm that chooses to raise its stock of productive capital should rent or buy additional space, buy and install new equipment, and train employees to use the extra equipment. In addition there are delivery lags and installation costs, making it more costly to adjust the capital stock quickly. All these costs are beyond the cost of buying additional capital goods.

• In addition, it is to be expected that these adjustment costs will be convex, i.e. they will depend on the size of the new investment. The higher the size of new investment, the greater will be the average adjustment cost of installing (or de-installing) an additional unit of capital.
The Jorgenson Model of the Flexible Accelerator

• Jorgenson (1963) assumed that, precisely because of the existence of adjustment costs, firms are not immediately but only gradually adjusting their stock of capital towards its “optimal” level, as determined by the user cost of capital and the marginal product of capital.

• He thus postulated an investment function which determined current investment as a fraction of the difference between the current and the “optimal” (desired) capital stock.

• However, Jorgenson did not derive the speed of adjustment, and thus the flow of investment, from a fully dynamic optimization problem. This was accomplished later by Lucas (1967), Gould (1968) and Treadway (1969), who, instead of postulating the investment function, as Jorgensen had done, solved for the optimal investment function from the dynamic problem of a firm maximizing the present value of its profits, subject to convex costs of adjusting its capital stock.

• Soon afterwards, Lucas and Prescott (1971) extended this framework to examine the determination of investment under uncertainty.
The $q$ Approach of Tobin: Market Value to Replacement Value of Installed Capital

- Tobin (1969) compared the ratio of the market value of installed capital of a firm, to the replacement cost of capital, naming this ratio $q$.

- Tobin argued that if the already installed capital stock of a firm has a higher value than the cost of replacing the capital goods that compose it, i.e. if $q$ is greater than one, then it will be profitable for the firm to invest, i.e. purchase and install new capital goods.

- Tobin argued that the rate of investment will be an increasing function of $q$, i.e. the ratio of the value of the already installed capital stock to its replacement cost (“Tobin’s $q$”).

- However, much like Jorgenson, Tobin did not derive his investment function from a dynamic optimization problem either.
The Abel and Hayashi Synthesis: Adjustment Costs and the Determination of $q$

• Abel (1982) and Hayashi (1982) showed that Tobin’s “q theory” and the theory of “adjustment costs” for investment of Jorgenson, as modeled by Lucas, Prescott, Gould and Treadway, can be combined in a unified framework. This synthesis of the two theories is now considered as the main neoclassical dynamic model of investment.

• The firm does not choose the level of its capital stock, by equating at any time the marginal product of capital to the sum of the real interest rate and the depreciation rate, but it chooses the amount of investment, taking into account the adjustment costs of the capital stock. Since marginal adjustment costs increase with the amount of investment, investment results in a gradual adjustment of the capital stock towards its steady state value.

• On the adjustment path, the firm takes into account both the current and future effects of its investment decisions. Thus, investment depends on both current and expected future developments in the value of the marginal product of capital and the user cost of capital.
A Simple Dynamic Model of Investment in the Presence of Convex Adjustment Costs

We consider a competitive firm producing a good $Y$. The production function of the firm is given by,

$$Y(t) = AF(K(t))$$

where $A$ is total factor productivity and $K$ the capital stock. The production function is characterized by diminishing returns. The market price of output and the capital stock is equal to unity.
A Simple Dynamic Model of the Determination of Investment

In order to change its capital stock, the firm must undertake gross investment \( I \). The change in its capital stock is thus determined by,

\[
\dot{I}(t) = \dot{K}(t) + \delta K(t)
\]

where \( \delta > 0 \) is a constant depreciation rate.

We assume that the cost of gross investment for the firm is equal to,

\[
I(t) + \psi(I(t))
\]

where \( \psi \) is a convex function, for which \( \psi(0)=0, \psi'>0 \) και \( \psi'' \geq 0 \).
The Installation Cost of Investment
The Optimal Choice of Investment

Assume that at time 0 the firm chooses an investment path that maximizes the present value of current and future profits. Assuming an infinite time horizon, the present value of the profits of the firm is equal to,

\[ V(0) = \int_{t=0}^{\infty} e^{-rt} \left( Y(t) - I(t) - \psi(I(t)) \right) dt \]

\( r \) is the real interest rate, assumed exogenous and constant. The present value is maximized under the constraint of the production function and the investment relation which links gross investment to the accumulation of capital. The current value Hamiltonian is defined by,

\[ \left( AF(K(t)) - I(t) - \psi(I(t)) \right) + q(t)(I(t) - \delta K(t)) \]

where \( q(t) \) is the multiplier of the capital accumulation constraint. \( q(t) \) is the shadow value of an additional unit of capital at \( t \).
First Order Conditions for the Optimal Choice of Investment

\[ q(t) = 1 + \psi'(I(t)) = 1 + \psi' \left( \dot{K}(t) + \delta K(t) \right) \]

**Interpretation:** the shadow value of an additional unit of capital \( q \) is equal to the marginal cost of investment. This is equal to the purchase price of capital goods (assumed equal to unity), plus the marginal installation cost \( \psi'(I(t)) \).

\[
\left( r + \delta - \frac{\dot{q}(t)}{q(t)} \right) q(t) = A \frac{\partial F(K(t))}{\partial K(t)} = AF_K(K(t))
\]

**Interpretation:** The firm will invest until the user cost of capital (on the left hand side) is equal to the marginal product of capital (on the right hand side). The user cost of capital is the real interest rate, plus the depreciation rate, minus the expected appreciation rate of the capital stock, multiplied by the shadow value of capital.
The Case of Zero Adjustment Costs

\[ q(t) = 1 \]

\[ \text{AF}_K(K(t)) = r + \delta \]

These are the usual first order conditions we have utilized so far.

The variables \( q \) and \( K \) jump immediately to their equilibrium values.

The shadow value of capital is continuously equal to unity, i.e. the purchase price of capital goods, and the capital stock adjusts immediately to the level where the marginal product of capital is equal to the real interest rate plus the depreciation rate, \( r + \delta \).

There is no investment flow, as the capital stock adjusts immediately. Without adjustment costs, this model does not determine gross investment, but only the equilibrium capital stock.

Prof. George Alogoskoufis, *Dynamic Macroeconomic Theory*, 2015
The Investment Function in the Presence of Adjustment Costs

In the general case, where there is a strictly convex adjustment cost function for the capital stock, investment is a positive function of \( q-1 \). Solving the first order condition for \( I \), we get that,

\[
I(t) = (\psi')^{-1} (q(t) - 1)
\]

Gross investment depends only on the difference of the shadow price of installed capital \( q \) from unity, as assumed by Tobin.

This dependence is positive because the marginal cost of investment is positive.

For this reason, the theory that depends on a rising adjustment cost on investment is referred to as the \( q \) theory of investment.
The Determinants of Marginal $q$

$q$ is equal to the marginal cost of investment. We have already used this condition to derive the investment function. To analyze the determinants of $q$, we must look into the second first order condition for the user cost of capital. This can be re-written as,

$$ q(t) = (r + \delta)q(t) - AF_K(K(t)) $$

This is a first order linear differential equation with variable coefficients, whose solution is,

$$ q(t) = \int_{s=t}^{\infty} e^{-(r+\delta)(s-t)} AF_K(K(s))ds $$

$q$ is the present value of all future marginal products of capital. As a result, $q$ depends negatively on the real interest rate and the depreciation rate, as well as factors that reduce the marginal product of capital, such as the capital stock. $q$ depends positively on factors that increase the marginal product of capital, such as total factor productivity.
The Dynamic Evolution of $q$ and the Capital Stock $K$

The determination of the shadow price of an additional unit of capital $q$, and the stock of capital $K$, can be inferred from the two first order conditions. Since both of these are non linear differential equations are nonlinear, their solution can be described by a phase diagram.

For a constant capital stock $K$ the first of the conditions implies that,

$$q = 1 + \psi'(\delta K)$$

For a constant shadow price of capital $q$, the second of the conditions implies that

$$q = \frac{1}{r + \delta} AF_K(K)$$

The steady state is determined at the intersection of the two curves. This steady state equilibrium is a saddle point, since $q$ is a non predetermined variable and $K$ is a predetermined variable. The adjustment path is unique.
The Dynamic Adjustment of $q$ and the Capital Stock $K$
Effects of a Permanent Increase in the Real Interest Rate

\[ \dot{q} = 0 \]

\[ \dot{K} = 0 \]

\[ q_{E'} \]

\[ q_E \]

\[ K_{E'} \]

\[ K_E \]
Effects of a Permanent Increase in Total Factor Productivity

![Graph showing the effects of a permanent increase in total factor productivity.](image)
Investment and Adjustment Costs for the Capital Stock

- We have analyzed the basic neoclassical model of investment with convex adjustment costs of investment.

- This model can be generalized so that the adjustment cost function depends not only on gross investment, but also on the stock of capital. It can also be generalized to simultaneously analyze investment and labor demand.

- It can also be generalized to allow for product market imperfections.

- Finally, it can be generalized to the case of uncertainty (Lucas and Prescott 1971).
The Appropriate Discount Rate for Choosing Optimal Investment under Uncertainty

• Just as under certainty, we shall assume that the firm chooses its investment path in order to maximize its value to its owners. The value is equal to the present value of the profits that the firm generates.

• Whereas under certainty the problem of the maximization of the present value of the profits of the firm is easily defined, under uncertainty, the question that arises is what should be the discount rate at which firms should discount future profits.

• In practice, it is often assumed that firms maximize the present discounted value of profits by using a deterministic discount rate, but this can only be justified under very specific assumptions.
Maximizing the Value of a Firm under Uncertainty

Let us assume that $V_t$ is the value of the firm in period $t$, and $\Pi_t$ is its per period revenue, net of investment expenditures. Then the rate of return $1+\pi_t$ from holding the firm for one period, will be given by,

$$1 + \pi_t = \frac{V_{t+1} + \Pi_{t+1}}{V_t}$$

For a consumer that invests in the firm under uncertainty, the rate of return from holding the firm $1+\pi_t$, and hence $V_t$ and $\Pi_t$ must satisfy,

$$u'(C_t) = \frac{1}{1+\rho} E_t \left[ (1+\pi_t) u'(C_{t+1}) \right] = \frac{1}{1+\rho} E_t \left[ \left( \frac{V_{t+1} + \Pi_t}{V_t} \right) u'(C_{t+1}) \right]$$
The Appropriate Discount Factor and the Marginal Utility of Consumption

- This condition is the same as the first order condition for investing in a “risky” asset, in the consumer problem under uncertainty.

- The returns generated by the firm in each state of nature are weighted by the marginal utility of consumption in that state.

- Thus, the discount factor that must be applied must take into account the correlation of the firm’s profits with the marginal utility of consumption at each state.
The Value of the Firm under Uncertainty

Solving the first order condition recursively forward, assuming away bubbles, we get the “fundamental solution” for the value of the firm $V_t$ as,

$$V_t = E_t \left( \sum_{s=1}^{\infty} \left( \frac{1}{1 + \rho} \right)^s \frac{u'(C_{t+s})}{u'(C_t)} \Pi_{t+s} \right)$$

The value of the firm is equal to the present discounted value of expected future profits. The discount rate for each period and for each state of nature is the marginal rate of substitution between consumption at time $t$ and consumption at that period and that state of nature. The higher the correlation between a firm’s profits and consumption, the higher will be the discount factor applied, and the lower the value of the firm.
Maximizing the Value of a Firm Using a Deterministic Discount Rate

In practice, it is often assumed that firms maximize the present discounted value of profits by using a deterministic discount rate. In this case,

\[ V_t = E_t \left( \sum_{s=1}^{\infty} \left( \Pi_{z=1}^{s} \frac{1}{1 + r_{t+z}} \right) \Pi_{t+s} \right) \]

where \( r_{t+z} \) is a deterministic interest rate in period \( t+z \). Although widely used, a specification such as this is generally inappropriate, because it suggests that at each date, the same discount factor is used to evaluate returns in different states of nature. This can only be justified under very specific assumptions.
The Case of Risk Neutrality

One set of assumptions that can be used to justify it is the assumption of risk neutrality on the part of consumers. If consumers are risk neutral, so their utility is linear in consumption and their marginal utility of consumption is constant, then the discount rate is not only deterministic, but also constant, and equal to the pure rate of time preference $\rho$. Thus, in the case of risk neutrality the value of the firm simplifies further to,

$$V_t = E_t \left( \sum_{s=1}^{\infty} \left( \frac{1}{1 + \rho} \right)^s \Pi_{t+s} \right)$$
The Case of Independence between Investment Decisions and the Relative Distribution of Returns Across States of Nature

Another set of assumptions that can be used to justify a deterministic discount rate is to assume that investment decisions do not affect the relative distribution of returns across states of nature, but only the scale of the firm. In this case the firm can use a constant discount rate, equal to the risk free rate, plus a risk premium that reflects the specific risk associated with the firm’s activities.

Both sets of assumptions are unlikely to hold in general, but they are often used as convenient approximations. It is worth noting however that they are good approximations only when considerations of risk aversion are not central to the problem analyzed.
The Lucas and Prescott Model of Investment under Uncertainty

We next turn to an examination of the investment decisions of a competitive firm under uncertainty, assuming that the objective of the firm is to maximize value with a deterministic discount rate.

The model we analyze is a linear quadratic variant of the class of models introduced by Lucas and Prescott (1971), and is similar in many respects to the q model we analyzed under certainty.

We shall use this model to derive aggregate investment in a rational expectations equilibrium.
The Investment Decision of a Competitive Firm

We assume a competitive firm $i$ that takes market prices as given. Its profit in period $t$ is defined by,

$$\Pi_{it} = \left( p_t Y_{it} - I_{it} - \frac{\psi}{2} (I_{it})^2 \right)$$

Output is produced using capital, through a linear production function of the form,

$$Y_{it} = AK_{it}$$

The evolution of the capital stock is determined by,

$$K_{it+1} = I_{it} + (1 - \delta)K_{it}$$
The Industry and the Representative Firm

\( i \) is continuous in the interval \([0,1]\). Industry output is thus given by,

\[
Y_t = \int_{i=0}^{1} Y_{it} di
\]

Since all firms face the same technology and the same market prices, the output of all firms will be the same. Thus, from now on we treat \( Y \) as the output of the representative firm. The same goes for all other variables, such as investment and the capital stock.
The Problem of the Representative Firm

The representative firm maximizes its present value,

\[ V_t = E_t \left( \sum_{s=1}^{\infty} \left( \frac{1}{1+r} \right)^s \Pi_{t+s} \right) = E_t \left( \sum_{s=1}^{\infty} \left( \frac{1}{1+r} \right)^s \left( p_{t+s} AK_{t+s} - I_{t+s} - \frac{\psi}{2} (I_{t+s})^2 \right) \right) \]

subject to the sequence of accumulation equations of the form,

\[ K_{t+1} = I_t + (1 - \delta) K_t \]

The stochastic processes driving the market price, the relative price of capital goods and total factor productivity are taken as given.
The Largangian and the First Order Conditions

The Largangian for the problem of the representative firm,

\[
E_t \left( \sum_{s=1}^{\infty} \left( \frac{1}{1+r} \right)^s \left( \left( p_{t+s} AK_{t+s} - I_{t+s} - \frac{\psi}{2} (I_{it+s})^2 \right) + q_{t+s} \left( I_{t+s} + (1 - \delta) K_{t+s} - K_{t+s+1} \right) \right) \right)
\]

From the first order conditions for a maximum we get the two familiar first order conditions, with the familiar interpretations.

\[
q_t = 1 + \psi I_t
\]

\[
(1 + r)q_t - (1 - \delta) E_t q_{t+1} = E_t p_{t+1} A
\]
The Determination of $q$ and 
industry investment $I$

The two first order conditions imply that,

$$I_t = \frac{1}{\psi} (q_t - 1)$$
$$q_t = \frac{1}{1 + r} E_t \sum_{s=0}^{\infty} \left( \frac{1 - \delta}{1 + r} \right)^s p_{t+s+1} A$$

Investment depends positively on the difference of $q$ from unity, which is the purchase price of investment goods.

$q$ turns out to be the discounted value of all expected future values of the marginal product of capital. It depends positively on the expected future evolution of the relative price for the product of the firm and the marginal productivity of capital $A$. It also depends negatively on the discount rate $r$ and the depreciation rate $\delta$. 

Prof George Alogoskoufis, Dynamic Macroeconomic Theory, 2015
The Determinants of Investment and the Capital Stock

Substituting for $q$, investment thus depends positively on the discounted value of all expected future changes in the value of the marginal product of capital, and negatively on the real interest rate, the depreciation rate and the adjustment cost parameter $\psi$.

\[
I_t = \frac{1}{\psi} \left( \frac{1}{1+r} \left( E_t \sum_{s=0}^{\infty} \left( \frac{1-\delta}{1+r} \right)^s p_{t+s+1} A \right) - 1 \right)
\]

From the definition of investment, the capital stock thus evolves according to,

\[
K_{t+1} = (1-\delta)K_t + \frac{1}{\psi} \left( \frac{1}{1+r} \left( E_t \sum_{s=0}^{\infty} \left( \frac{1-\delta}{1+r} \right)^s p_{t+s+1} A \right) - 1 \right)
\]
From Industry Demand to Equilibrium Prices

Although for each competitive firm the price of output is taken as given, for the industry, the market price will be determined endogenously, from the equation of total demand for its product and industry supply. Industry supply will depend on investment and the evolution of the capital stock. This allows us to solve for the equilibrium price endogenously, as a function of the capital stock, and characterizing the evolution of the capital stock and the equilibrium price in a rational expectations equilibrium.

Assume that industry demand is linear in the price and given by,

\[ Y_t = D - bp_t + v_t \]

Then, the output price is determined by,

\[ p_t = \frac{1}{b}(D - Y_t + v_t) = \frac{1}{b}(D - AK_t + v_t) \]
The Determinants of Investment in Rational Expectations Equilibrium

We can use the price equation to substitute for the expected equilibrium price in the capital accumulation equation, and solve for the evolution of the capital stock as a function of only exogenous shocks. This results in,

\[ K_{t+1} = \lambda K_t + \frac{\lambda}{\psi(1-\delta)(1+r-\lambda)} \left( \frac{AD}{b} - (r + \delta) \right) + A \frac{\lambda}{b \psi(1-\delta)(1+r)} E_t \sum_{s=0}^{\infty} \left( \frac{\lambda}{1+r} \right)^s v_{t+s+1} \]

where, \( \lambda < 1 \), and,

\[ \lambda + \mu = \frac{\psi(1+r)(1+(1-\delta)^2) + A^2}{\psi(1+r)(1-\delta)} > 2 \]

\[ \lambda \mu = 1 + r \]
The Determinants of Equilibrium Investment and the Capital Stock

The evolution of the industry capital stock in rational expectations equilibrium depends on current expectations about the whole future path of disturbances to industry demand $v$, and parameters such as the discount rate $r$, the productivity of capital $A$, the adjustment cost parameter $\psi$, the depreciation rate $\delta$, the size of the market $D$ and the price responsiveness of industry demand $b$.

$1-\lambda$, the speed of adjustment, depends only on the discount rate and technological parameters.
The Structure of the Lucas and Prescott Model

- The representative firm chooses investment, and implicitly the capital stock and output, to maximize the present value of its profits, taking as given the market price of its output, the exogenous relative price of capital goods and exogenous productivity $A$.

- In order to compute the full equilibrium, once investment and output of the representative firm are determined, industry output is replaced in the industry demand function to solve for the equilibrium price in terms of the exogenous stochastic process driving industry demand.

- The full equilibrium is thus described by a pair of interrelated capital accumulation and price equations, which are consistent with continuous market clearing.