Dynamic Stochastic Models

Dynamic stochastic models, and an appropriate expectations hypothesis are indispensable if one were to model conditions in which there is uncertainty about the future.

Unlike the deterministic models with perfect foresight which we have used so far, in which there was no uncertainty about the future, a more realistic treatment of dynamic macroeconomics requires the modeling of uncertainty and uncertain expectations about the future.

Thus, models need not only be dynamic, but also stochastic, in the sense that they encompass uncertainty. In this lecture we introduce the basic concepts and discuss the properties and alternative solution methods for such dynamic stochastic models under rational expectations.

Dynamic stochastic models can be contrasted with dynamic deterministic models which do not allow for uncertainty. A dynamic deterministic model consists of a set of differential or difference equations that describe exactly how the system will evolve over time. Thus, there is no uncertainty about the evolution of such a model economy, unless the model is characterized by multiple equilibria.

A static stochastic model allows for uncertainty and describes the interactions of random variables. A dynamic stochastic model describes the dynamic interactions among stochastic processes, which constitute the dynamic equivalents of random variables. If a dynamic stochastic model is solved several times, it will not produce identical results, because of differences in its stochastic elements. Different runs of a dynamic stochastic model are different realizations of a stochastic process, and would in general result in different outcomes. Stochastic models embody both randomness and uncertainty.

Thus, instead of describing a process which can only evolve in one way, as in the case of solutions of deterministic systems of ordinary differential or difference equations, in a dynamic stochastic model there is indeterminacy due to randomness. Even if the initial conditions (or starting points) are known, there are several, often infinitely many, directions in which the process may evolve over time.
A Stochastic Expectational Model of a Competitive Market

For illustrative purposes, we introduce a simple expectational model of a competitive market, in order to illustrate the differences between deterministic and stochastic models, and highlight the importance of the treatment of expectations. This is a log-linear model in which demand is a function of the current price, but in which supply decisions are made one period in advance, based on the prior expectations of firms for the current price. Hence, the expectations about the current price, based on information available one period in advance, partly determine the supply of the commodity in question. In addition, there is a stochastic supply shock, which cannot be known in advance. Ex post, the market clears at a price which equates this, partly predetermined, supply with current market demand.

This simple expectational model, known as the cobweb model, because of its dynamic properties under adaptive expectations, has played a very important role in the study of market cycles and expectations in economics.
The Stochastic Expectational Competitive Market Model

The model consists of log-linear demand and supply functions, and an equilibrium condition requiring that the current price adjusts to equate demand and supply.

The supply of the commodity is question is determined by,

\[ q_t^s = a + b p_t^e + v_t \]

where \( q_t^s \) denotes the logarithm of the supply of commodity \( Q \) in period \( t \), and \( p_t^e \) the expectation of suppliers about the logarithm of the market price \( P \) at \( t \), based on information they possess in the previous period \( t - 1 \). \( a \) and \( b \) are constant positive parameters, and \( v_t \) is a stochastic supply shock. It is assumed that suppliers have to make their supply decisions before the current market price is known, and that supply is a positive function of the expected price. \( b \) is the elasticity of supply with respect to the expected price.

The demand for the commodity is determined by,

\[ q_t^d = c - d p_t \]

where \( q_t^d \) denotes the logarithm of the demand for commodity \( Q \) in period \( t \), and \( p_t \) is the logarithm of the market price \( P \) at \( t \). \( c \) and \( d \) are constant positive parameters. It is assumed that demand is a negative function of the current price. \( d \) is the elasticity of demand with respect to the current price.

The market being competitive, equilibrium is determined by the equality of demand and supply. Thus, the equilibrium condition is that,

\[ q_t^s = q_t^d = q_t \]

where \( q_t \) is the equilibrium quantity produced.
Equilibrium in the Stochastic Expectational Competitive Market Model

Solving the model for the equilibrium price, we get,

\[
p_t = \frac{c - a}{d} - \frac{b}{d} p_t^e - \frac{1}{d} v_t
\]

The equilibrium price depends negatively on the prior expectation of the equilibrium price by suppliers. The higher the expected price, the higher the supply, and therefore the lower the resulting equilibrium price. In order to solve for the market equilibrium in terms of the market fundamentals, like the parameters \(a, b, c, d\) and the supply shock \(v_t\), we must make an assumption about the formation of expectations.
Equilibrium in the Absence of Uncertainty and Perfect Foresight

If there is no uncertainty, and suppliers know the supply shock with certainty, a natural assumption about the formation of expectations in this case is the assumption of perfect foresight. In fact, this is the assumption we have been utilizing in the optimizing deterministic models of growth we have analysed so far.

Since everybody knows everything with certainty, suppliers can form a perfect prediction of the equilibrium price, and in this case $p^e_t = p_t$. Hence, equilibrium prices and quantities under perfect foresight are given by,

$$p_t = \frac{c - a}{b + d} - \frac{1}{b + d} v_t$$

$$q_t = \frac{cb + da}{b + d} + \frac{d}{b + d} v_t$$

Since there is no uncertainty about either the parameters $a, b, c, d$ or the supply shock $v_t$, suppliers can forecast the equilibrium price perfectly, and the model becomes equivalent to a simple demand and supply competitive model, in which the price moves immediately to clear the market.

If $v_t$ is a random variable, or a stochastic process, the model is stochastic, and prices and quantities fluctuate randomly, as they are driven by the stochastic process followed by the exogenous supply shock $v_t$. Otherwise, the model is deterministic.
Uncertainty and Adaptive Expectations

What if there is uncertainty about the evolution of the supply shock $v_t$? Then suppliers, not knowing the market price, which will depend on the unknown supply shock, will have to form expectations about the price in advance.

The key hypothesis traditionally made about the formation of expectations under uncertainty was the hypothesis of adaptive expectations.

The adaptive expectations hypothesis was an empirically motivated short cut to the formation of expectations, postulating that agents adapt their expectations gradually to correct for past expectational errors. It was the key expectational hypothesis for many decades before the appearance of the rational expectations hypothesis.

In the context of our simple competitive model, the adaptive expectations hypothesis would postulate that supplier’s expectations about the market price evolve according to,

$$p_t^e - p_{t-1}^e = (1 - \lambda)(p_{t-1} - p_{t-1}^e)$$

This implies that if the equilibrium price in the previous period was higher than the previously expected equilibrium price, the expected equilibrium price for next period will be revised upwards. $0 \leq \lambda \leq 1$ measures the degree of inertia in the revision of expectations. If $\lambda = 1$ there is full inertia, and expectations are never revised. This is the case of static expectations, implying that $p_t^e = p_{t-1}^e$ for all $t$. If $\lambda = 0$ there is no inertia, and expectations are fully revised, and are therefore equal to the previous price. In the context of our model, this case of fully adaptive expectations implies that,

$$p_t^e = p_{t-1}$$
Equilibrium under Adaptive Expectations

Substituting the fully adaptive expectations hypothesis for the expected price in the expectational equilibrium price equation of the competitive market model, we get that the equilibrium price is determined by,

$$p_t = \frac{c - a}{d} - \frac{b}{d} p_{t-1} - \frac{1}{d} v_t$$

This is a stochastic first order difference equation which determines the equilibrium price as a negative function of the previous equilibrium price. In the absence of shocks the equilibrium price will display oscillations. High prices today cause an increase in supply and low prices tomorrow, and vice versa. It was this type of behavior of prices (and quantities) that led Kaldor (1934) to name this model the cobweb model, as the diagrammatic representation of this type of price and quantity adjustment looks like a cobweb.

In the absence of supply shocks, the oscillations will lead to convergence to an equilibrium price only if the root of the difference equation is less than unity in absolute terms, i.e.

$$\left| \frac{b}{d} \right| < 1$$

This requires that the elasticity of supply with respect to the expected price $b$ is lower (in absolute terms) than the elasticity of demand with respect to the current price $d$. If the elasticities are equal, there will be non convergent oscillations, and if the supply elasticity exceeds the demand elasticity there will be divergent oscillations, and the model would be unstable.

In general, in the presence of supply shocks, and if the stability condition is satisfied, the solution of the difference equation takes the form,

$$p_t = \frac{c - a}{b + d} - \frac{1}{d} \sum_{i=0}^{\infty} \left( -\frac{b}{d} \right)^i v_{t-i}$$

Equilibrium prices (and quantities) are a geometric distributed lag of current and past supply shocks, with alternating signs. If the supply shocks are stochastic, equilibrium prices and quantities will be stochastic as well. The solution is backward looking.
The Cobweb Model under Adaptive Expectations
The Rational Expectations Hypothesis

The rational expectations hypothesis was introduced by Muth (1961), who illustrated its significance using a version of this particular expectational market model. Muth (1961) argued that,

“expectations, since they are informed predictions of future events, are essentially the same as the predictions of the relevant economic theory. ... we call such expectations "rational". ... The hypothesis can be rephrased a little more precisely as follows: that expectations of firms (or, more generally, the subjective probability distribution of outcomes) tend to be distributed, for the same information set, about the prediction of the theory (or the "objective" probability distributions of outcomes).” (p. 316).

Rational expectations are thus defined as the mathematical expectations for the future evolution of a stochastic variable, based on the model determining this variable and a given information set.

Thus defined, the rational expectation of a variable \( x \) in period \( t + 1 \), based on available information in period \( t \), is the mathematical expectation of the variable at \( t + 1 \), conditional on the information set available at \( t \).

\[
E_t x_{t+1} = E(x_{t+1} | I_t)
\]

where \( I_t = \{ x_{t-i}, z_{t-i}, i = 0,1,2,\ldots,\infty \} \) is the set of available information, which consists of current and past values of the variable \( x \) itself, as well as the current and past values of a set of variables \( z \), which affect, and thus can help predict, the future values of \( x \). It is worth noting that this definition of the information set does not involve loss of memory, as whatever is known in period \( t \) is also known in period \( t + 1 \) and all future periods.

More generally, we define the rational expectation of a variable \( x \) at \( t + s \), based on available information at \( t \), as,

\[
E_t x_{t+s} = E(x_{t+s} | I_t)
\]

In what follows we shall simplify the notation by using \( E_t x_{t+1} \) in place of \( E(x_{t+1} | I_t) \). Note that in order to define rational expectations for a variable more precisely, it is not enough to specify the information set. One also needs to specify the model determining how this variable evolves over time.
The Rational Expectations Hypothesis in the Competitive Market Model

Imposing the rational expectations hypothesis in our competitive market model, implies that the formation of expectations about prices will take account of the price determination process itself. Hence, expectations will be formed taking into account the equilibrium price determination equation.

\[ p_t = \frac{c - a}{d} - \frac{b}{d} p_t^e - \frac{1}{d} v_t \]

Applying the rational expectations hypothesis to this equation, by taking the rational expectation of the market price conditional on information available up to period \( t - 1 \), implies that,

\[ p_t^e = E_{t-1} p_t = \frac{c - a}{d + b} - \frac{1}{d + b} E_{t-1} v_t \]

Hence, the rational expectation of the equilibrium price is equal to the equilibrium price in the absence of supply shocks, adjusted for the rational expectation of the impact of the supply shock, based on information available in period \( t - 1 \).

Substituting for this rational expectation in the expectational price equation, we get that the equilibrium price is given by,

\[ p_t = \frac{c - a}{b + d} - \frac{1}{d(b + d)} (b(v_t - E_{t-1} v_t) + d v_t) \]

Supply shocks depress the equilibrium price because they increase supply relative to demand. However, to the extent that they are unexpected, there is an extra depressing effect, because suppliers have overestimated the equilibrium price, and hence have increased supply by more than they would have done otherwise. In the case where supply shocks are fully expected, there are no unanticipated shocks, and we are back to the perfect foresight case. In the case where supply shocks are fully unexpected, the rational expectations solution takes the form,

\[ p_t = \frac{c - a}{b + d} - \frac{1}{d} v_t \]
Adaptive versus Rational Expectations

The rational expectations solution in general differs from the solution under adaptive expectations, as it does not depend on the whole history of past supply shocks, but only on whatever information is required at $t-1$ to form a rational expectation of the supply shock in period $t$. In addition, there is no cobweb under rational expectations, as expectations about current prices do not depend only on past prices as postulated by the adaptive expectations hypothesis. However, in order to fully solve the model under rational expectations we need to assume a model for the evolution of the exogenous supply shock $v_t$. This model will be the one used to calculate the rational expectation of prices on the basis of past information.

We have already solved the model for perfectly anticipated and perfectly unanticipated supply shocks. In the case of partly anticipated supply shocks, one has to make specific assumptions about the stochastic process determining the evolution of such shocks. We shall postpone this discussion until we examine how rational expectations of a stochastic process are computed. We thus turn to the discussion of solution methods for more general stochastic models of exogenous shocks and more general dynamic stochastic models under rational expectations.
Rational Expectations for Stationary Linear Autoregressive Processes

We begin with the method for computing the rational expectations solution for the simplest dynamic stochastic model, that of a univariate stochastic process. Exogenous shocks in macroeconomics are often assumed to follow such processes.

Let us assume a variable $x$, which follows a linear first order autoregressive stochastic process of the form,

$$x_t = (1 - \lambda)\bar{x} + \lambda x_{t-1} + \epsilon_t$$

where, $\bar{x}$ is a constant, the mean of the variable $x$, and $\epsilon$ a white noise stochastic process, with zero mean and constant variance.

We shall define the deviation of $x$ from its mean as,

$$\hat{x}_t = x_t - \bar{x}$$

It thus follows that,

$$\hat{x}_t = \lambda \hat{x}_{t-1} + \epsilon_t$$

It is easy to see, by repeated substitutions, that,

$$E_t \hat{x}_{t+1} = \lambda \hat{x}_t, \quad E_t \hat{x}_{t+2} = \lambda^2 \hat{x}_t, \ldots, \quad E_t \hat{x}_{t+s} = \lambda^s \hat{x}_t$$

The rational expectation of a first order linear autoregressive stochastic process depends only on its current value, with a coefficient that depends only on $\lambda$.

If the stochastic process is stationary, i.e., if $-1 < \lambda < 1$, then the impact of the current value of the variable on its rational expectation at time $t + s$ is decreasing with $s$. As $s$ approaches infinity, the limit of the rational expectation is given by,

$$\lim_{s \to \infty} E_t \hat{x}_{t+s} = \lim_{s \to \infty} \lambda^s \hat{x}_t = 0, \quad \text{or} \quad \lim_{s \to \infty} E_t x_{t+s} = \bar{x}$$

The mean of the stationary stochastic process $x$, is the limit towards which rational expectations of its future evolution converge over time.
Rational Expectations for Non-Stationary Linear Autoregressive Processes

If the process is non-stationary, for example a random walk for which $\lambda = 1$, then rational expectations imply that,

\[
E_t x_{t+1} = x_t, \quad E_t x_{t+2} = x_t, \ldots, \quad E_t x_{t+s} = x_t
\]

In this case, the rational expectation for the future value of $x$ is the current value of $x$, independently of $s$, as the variable is a random walk and does not converge to a long run equilibrium.

These methods generalize to higher order stationary and non-stationary autoregressive moving average processes.
Rational Expectations Equilibrium in the Competitive Market Model

Returning to the expectational competitive market model, let us assume that the supply shock $v_t$ follows a first order autoregressive process of the form,

$$v_t = \lambda v_{t-1} + \epsilon_t$$

where, $0 < \lambda < 1$, and $\epsilon_t$ is a white noise process.

We have shown that the equilibrium price under rational expectations is given by,

$$p_t = \frac{c - a}{b + d} - \frac{1}{d(b + d)} \left( b(v_t - E_{t-1}v_t) + d v_t \right)$$

Since in this case $E_{t-1}v_t = \lambda v_{t-1}$, if follows that,

$$p_t = \frac{c - a}{b + d} - \frac{1}{d} v_t + \frac{b\lambda}{d(b + d)} v_{t-1}$$

The current supply shock depresses the equilibrium price, as it increases current supply, but the past supply shock boosts the price, because, by positively affecting prior expectations about the current supply shock, it induces firms to reduce supply, as it reduces their expectation about the current price. Note that the dynamics of the market price differ from the case of adaptive expectations and there is no cobweb pattern in this case either. Substituting for $v_t$ in the price equation we get,

$$p_t = \frac{c - a}{b + d} - \frac{b(1 - \lambda) + d}{d(b + d)} v_{t-1} - \frac{1}{d} \epsilon_t$$
First Order Linear Expectational Models

We now turn to the rational expectations solution of a linear stochastic model in which a variable $y$ depends on the rational expectation of its future value, and another exogenous stochastic variable $x$. The model is described by a first order linear expectational difference equation of the form,

$$y_t = aE_t y_{t+1} + bx_t$$

The rational expectations hypothesis implies that economic agents know that the variable $y$ is determined by this expectational difference equation. We also assume that all economic agents have access to the same information set.

There are a number of methods for the solution of such a model. All methods are based on the law of iterated expectations, which requires that the current expectation of the future expectation of a future value of a variable, is nothing more than the current expectation of the future value of the variable. That is, that,

$$E_t (E_{t+z} x_{t+s}) = E_t x_{t+s}$$

for $z, s \geq 1$, and $z \leq s$. 
The Method of Repeated Substitutions

The simplest method for solving this model is the method of repeated substitutions, a method that we also used in finding the rational expectations for the simple first order autoregressive process in the previous section.

Using the law of iterated expectations, we get that,

\[ E_t y_{t+1} = a E_t \left( E_{t+1} y_{t+2} \right) + b E_t x_{t+1} = a E_t y_{t+2} + b E_t x_{t+1} \]

Substituting in the original model, we get,

\[ y_t = a^2 E_t y_{t+2} + a b E_t x_{t+1} + b x_t \]

Repeatedly substituting the future expectations of \( y \), until \( T \) future periods, we get,

\[ y_t = a^{T+1} E_t y_{T+1} + b \sum_{s=0}^{T} a^s E_t x_{t+s} \]

In order to have convergence of the last term as \( T \) tends to infinity, the absolute value of \( a \) must be less than one, and the expected value of \( x \) should not increase too quickly. If the expected value of \( x \) increases exponentially, its growth rate should not exceed \((1/a) - 1\).

Under these conditions, it follows that,

\[ \lim_{T \to \infty} a^{T+1} E_t y_{T+1} = 0 \]

Then, a solution can be derived as,

\[ y_t = b \sum_{s=0}^{\infty} a^s E_t x_{t+s} \]

This satisfies the terminal condition and is thus a solution of the model. It suggests that the current value of the endogenous variable \( y \) is the discounted sum of the expected future values of the exogenous variable \( x \), with a discount factor equal to \( a \). This solution is usually called the fundamental solution.
Non-fundamental Solutions

It is however worth noting that the fundamental solution is not the only solution. The fundamental solution is based only on the minimum number of variables (\(x\) in our case), the so-called fundamentals, and satisfies the terminal condition. If the terminal condition is not satisfied, then there is a host of other non-fundamental solutions.

Suppose there is an alternative solution which involves an additional variable \(z\). This solution takes the form,

\[
y_t = b \sum_{s=0}^{\infty} a^s E_t x_{t+s} + z_t
\]

One can easily demonstrate that if the variable \(z\) satisfies,

\[
z_t = a E_t z_{t+1}
\]

or, equivalently,

\[
E_t z_{t+1} = \frac{z_t}{a}
\]

then this is also a solution of the model.

However, it is worth noting that because \(a < 1\), the mathematical expectation of the future \(z\) explodes over time. This can be proven by taking the limit of the mathematical expectation as time tends to infinity. This limit is given by,

\[
\lim_{s \to \infty} E_t z_{t+s} = \left(\frac{1}{a}\right)^s \begin{cases} z_t & \text{if } a > 1 \\ \pm \infty & \text{if } a < 0 \end{cases}
\]

depending on whether \(z\) is positive or negative.

Solutions based on non-fundamental variables such as \(z\) are called bubbles, as opposed to solutions which are based only on the fundamentals.
The Method of Factorization and Future Mathematical Expectations Operator

The factorization method requires the use of the future mathematical expectations operator $F$. We define the future mathematical expectations operator $F$ for a variable $x$, as,

$$F^s x_t = E_t x_{t+s}$$

for $s = \ldots, -1, 0, 1, 2, \ldots$. Note that this operator is conditional on the information set available in period $t$ which does not change when the operator is raised to different powers.

It follows that, for $s = 0$, $Fx_t = x_t$, for $s = 1$, $Fx_t = E_t x_{t+1}$, $F^2x_t = E_t x_{t+2}$, and so on so forth. In addition, for $s = -1$, $F^{-1}x_t = x_{t-1}$, for $s = -2$, $F^{-2}x_t = x_{t-2}$ and so on so forth. Thus, the future mathematical expectations operator is the inverse of the lag operator $L$, which we have used for the solution of difference equations.
The Factorization Method for a First Order Model

Our first-order expectational model takes the form,

$$ y_t = aE_t y_{t+1} + bx_t $$

Using the future mathematical expectations operator, and assuming that $-1 < a < 1$, this can be re-written as,

$$ y_t = aFy_t + bx_t = \frac{b}{1 - aF} x_t = b \sum_{s=0}^{\infty} a^s F^s x_t = b \sum_{s=0}^{\infty} a^s E_t x_{t+s} $$

The right hand side is the same as the fundamental solution that we found using the method of repeated substitutions. Hence, using the factorization method, one can arrive at the fundamental solution in a few simple steps.
The Method of Undetermined Coefficients

The method of undetermined coefficients consists in using a presumed functional form of a solution with undetermined coefficients, obtain the mathematical expectation of the presumed solution, replace this for the expectation in the original model, and compare the coefficients of the resulting equation with the undetermined coefficients of the presumed solution. If the form of the presumed solution is correct, this will suffice to determine the undetermined coefficients.

For example, if our guess is that the solution has the form,

\[ y_t = \sigma \sum_{s=0}^{\infty} \mu^s E_t x_{t+s} \]

where \( \sigma \) and \( \mu \) are undetermined coefficients, then it follows that,

\[ E_t y_{t+1} = \sigma \sum_{s=0}^{\infty} \mu^s E_t x_{t+s+1} \]

Substituting from the equation above in the original expectational model \( y_t = a E_t y_{t+1} + bx_t \), we get,

\[ y_t = a E_t y_{t+1} + bx_t = a \sigma \sum_{s=0}^{\infty} \mu^s E_t x_{t+s+1} + bx_t \]

Comparing coefficients between the equation above and our original guess in the first equation, we find that \( \sigma = b \) and \( \mu = a \). This confirms our presumption and determines the coefficients. The solution is exactly the same as with the two other methods.
The Determination of Stock Prices in Efficient Capital Markets

In order to see how these methods are applied, we shall use two simple economic models which result in equations of the form we have analyzed.

In our first model we assume a capital market in which investors are risk neutral. They choose between a common stock with an uncertain return, and a safe asset with a certain rate of return $r$.

In equilibrium, arbitrage will ensure that the expected rate of return of the stock will be equal to the rate of return of the safe asset.

$$\frac{E_t p_{t+1} - p_t}{p_t} + \frac{d_t}{p_t} = r$$

where $p$ is the price of the stock in the capital market, and $d$ is the dividend. The expected rate of return of the stock is equal to the expected capital gain, plus the dividend as a proportion of the stock price. In equilibrium in an efficient capital market this cannot differ from the rate of return of the safe asset $r$.

The above equilibrium condition can be rearranged as,

$$p_t = \frac{1}{1 + r} \left( E_t p_{t+1} + d_t \right)$$

This has the form of the first order linear expectational model, with $0 < a = b = 1/(1 + r) < 1$.

The fundamental solution of the model gives us the stock price as the discounted sum of the current and expected future dividends.

$$p_t = \frac{1}{1 + r} \sum_{s=0}^{\infty} \left( \frac{1}{1 + r} \right)^s E_t d_{t+s}$$

The stock price is the present value of the current and expected future dividends, discounted by the rate of return of the safe asset.
The Cagan Model of Money Demand

In our second model we assume consumers and firms which decide between holding money and goods in the face of inflation. Money is a nominal asset whose real value is affected negatively by inflation. In this case, the demand for money is a negative function of expected inflation, and money market equilibrium requires that,

\[
\frac{M_t}{P_t} = \exp \left( -\alpha \left( \frac{E_t P_{t+1} - P_t}{P_t} \right) \right)
\]

where \( M \) is the nominal money supply, \( P \) the price level, and \( \alpha > 0 \) the semi-elasticity of money demand with respect to expected inflation. This model was first used by Cagan (1956) in order to explain hyper-inflations.

Taking logarithms on both sides, and denoting the logarithm of the nominal money supply by \( m \) and the logarithm of the price level by \( p \), the model can be written in log-linear form as,

\[
m_t - p_t = -\alpha (E_t P_{t+1} - P_t)
\]

In log-linearising the model we have used the approximation that, for a small \( x \), \( \ln(1 + x) \) is approximately equal to \( x \). Solving for \( p_t \) we get,

\[
p_t = \frac{\alpha}{1 + \alpha} E_t P_{t+1} + \frac{1}{1 + \alpha} m_t
\]

The equation above has the form of the first order linear expectational model, with \( a = \frac{\alpha}{1 + \alpha} \), \( b = 1 - a \). The fundamental solution takes the form,

\[
p_t = \frac{1}{1 + \alpha} \sum_{s=0}^{\infty} \left( \frac{\alpha}{1 + \alpha} \right)^s E_t m_{t+s}
\]

The current price level depends on the discounted expectations about the evolution of the money supply in the future, with a discount factor equal to \( 0 < \alpha/(1 + \alpha) < 1 \). Thus, it is the full expected future path of the money supply that determines the current price level in this model.
Second-Order Linear Expectational Models

We next turn to the methods of solving a second order, dynamic stochastic linear model under rational expectations. In this model, we assume that an endogenous variable $y$ depends on the rational expectation of its future value $y_{t+1}$, an exogenous stochastic variable $x$, but also its own lagged value $y_{t-1}$.

This model combines rational expectations about the future value of the variable, with the impact of lagged values of the variable, and, as we shall see in future lectures often arises in macroeconomics. Our model is linear and takes the form,

$$y_t = aE_t y_{t+1} + by_{t-1} + cx_t$$

where, $a, b > 0$, and $a + b < 1$.

This model, which is essentially a second order difference equation, can be solved using either the factorization method, or the method of undetermined coefficients. Obviously, both methods result in the same fundamental solution. The method of repeated substitutions would be too cumbersome for this model.
The Factorization Method

Using the future expectations operator $F$, the second-order linear expectational model can be written as,

$$y_t = aFy_t + bF^{-1}y_t + cx_t$$

where $F^{-1}$, the inverse of the future expectations operator, is the same as the lag operator.

Moving all the terms that contain $y$ on the left-hand side, we get,

$$y_t (1 - aF - bF^{-1}) = cx_t$$

The left hand side of the equation above can be multiplied and divided by $-F/a$, which results in,

$$-aF^{-1}y_t \left( F^2 - \frac{1}{a}F + \frac{b}{a} \right) = cx_t$$

This can be rewritten as,

$$y_t \left( F^2 - \frac{1}{a}F + \frac{b}{a} \right) = -\frac{c}{a}Fx_t$$

The equation above can be factorized as,

$$y_t(F - \lambda)(F - \mu) = y_t \left( F^2 - (\lambda + \mu)F + \lambda \mu \right) = -\frac{c}{a}Fx_t$$

where $\lambda$ and $\mu$ are the two roots of the characteristic polynomial of,

$$\left( F^2 - \frac{1}{a}F + \frac{b}{a} \right)$$

we know that, $\lambda + \mu = 1/a$, $\lambda \mu = b/a$. It is simple to show that one root is smaller than unity and the other is higher than unity.
Characterizing the Size of the Roots

The characteristic polynomial of the model is given by,

\[ \Phi(\phi) = \left( \phi^2 - \frac{1}{a}\phi + \frac{b}{a} \right) \]

In order to show that one root is smaller than unity, we shall calculate the characteristic polynomial for \( \phi = 0 \) and \( \phi = 1 \). We get,

\[ \Phi(0) = \frac{b}{a} > 0, \Phi(1) = -\frac{1 - a - b}{a} < 0 \]

Hence, there is one root, let us assume it is \( \lambda \), which lies between zero and one, and for which \( \Phi(\lambda) = 0 \).

The second root \( \mu \) is determined by,

\[ \mu = \frac{b}{a\lambda} \]

We shall have \( \mu > 1 \), if \( \lambda < (b/a) \). This is actually true, since,

\[ \Phi \left( \frac{b}{a} \right) = -\frac{b(1 - a - b)}{a^2} < 0 \]

Therefore we shall have that \( \lambda < (b/a) < 1 \) and \( \mu > 1 \).
Solution of the Second-Order Linear Expectational Model

We have shown that the second-order linear expectational model can be factorized as,

\[ y_t(F - \lambda)(F - \mu) = -\frac{c}{a}Fx_t \]

where \( \lambda < (b/a) < 1 \) and \( \mu > 1 \) are the two roots of the characteristic equation of the model.

The model is saddlepath stable, as it has one predetermined variable, \( y_{t-1} \) and one non-predetermined variable, the rational expectation \( E_t(y_{t+1}) \). Dividing both sides by \( F(\mu - F) \), we get,

\[ y_t \left( 1 - \lambda F^{-1} \right) = \frac{c}{a} \frac{1}{\mu - F}x_t = \frac{c}{a\mu} \frac{1}{1 - \mu^{-1}F}x_t = \frac{\lambda c}{b} \frac{1}{1 - \mu^{-1}F}x_t \]

In the final expression on the RHS we have made use of the property that \( a\mu = b/\lambda \). It thus follows that,

\[ y_t = \lambda y_{t-1} + \frac{\lambda c}{b} \frac{1}{1 - \mu^{-1}F}x_t = \lambda y_{t-1} + \frac{\lambda c}{b} \sum_{s=0}^{\infty} \left( \frac{1}{\mu} \right)^s E_t x_{t+s} \]

This is the fundamental solution of the second-order linear expectational model. The current value of the endogenous variable \( y \) depends on the discounted sum of the expected future values of the exogenous variable \( x \), with a discount factor equal to \( 1/\mu < 1 \). It also depends on its own lagged value, with a coefficient equal to \( \lambda < 1 \).
The Method of Undetermined Coefficients

In order to apply the method of undetermined coefficients, as in the case of the first order model, we presume that the solution of the model takes the form,

\[ y_t = \phi y_{t-1} + \psi \sum_{s=0}^{\infty} \omega^s E_t x_{t+s} \]

with undetermined coefficients \( \phi, \psi, \omega \). The rational expectation of the future \( y \) is given by,

\[ E_t y_{t+1} = \phi y_t + \psi \sum_{s=1}^{\infty} \omega^{s-1} E_t x_{t+s} \]

Substituting this in the original model we get,

\[ y_t = \frac{b}{1 - a \phi} y_{t-1} + \frac{c}{1 - a \phi} x_t + \frac{a \psi}{1 - a \phi} \sum_{s=1}^{\infty} \omega^{s-1} E_t x_{t+s} \]

Comparing coefficients between the equation above and the presumed solution we can solve for the undetermined coefficients, as,

\[ \phi = \frac{b}{1 - a \phi}, \quad \psi = \frac{c}{1 - a \phi}, \quad \omega = \frac{a}{1 - a \phi} \]

From the first equation above, \( \phi \) will be the smaller root of the polynomial,

\[ \Phi(\phi) = \phi^2 - \frac{1}{a} \phi + \frac{b}{a} \]

This is none other than the characteristic polynomial analyzed in the factorization method. There will be two roots, \( 0 < \lambda < 1 \) and \( \mu > 1 \).

The roots will satisfy, \( \lambda + \mu = \frac{1}{a}, \lambda \mu = \frac{b}{a} \) and the undetermined coefficients will be defined as \( \phi = \lambda < 1, \psi = \frac{\lambda c}{b} \) and \( \omega = \frac{1}{\mu} < 1 \).
A Second-Order Model of the Capital Market

Assume an efficient capital market in which risk-neutral investors choose between a common stock and a safe asset with a rate of return $r$. In equilibrium, arbitrage will ensure that the expected rate of return of the stock will be equal to the rate of return of the safe asset.

\[
\frac{E_t p_{t+1} - p_t}{p_t} + \frac{d_t}{p_t} = r,
\]

where $p$ is the price of the stock, and $d$ is the dividend. The expected rate of return of the stock is equal to the expected capital gain, plus the dividend as a proportion of the stock price.

The firm has a known dividend policy of the form,

\[
d_t = d_0 + \delta p_{t-1}
\]

where $0 < \delta < r$. Dividends per share are a constant $d_0$ plus a percentage of the price of the share in the previous period. Presumably this is a policy adopted by the firm with the intention to boost the price of the stock.

Substituting the dividend policy into the arbitrage equation and solving for the price, we get,

\[
p_t = \frac{1}{1 + r} \left( E_t p_{t+1} + \delta p_{t-1} + d_0 \right)
\]

Because of the dependence of current dividends on the stock price of the previous period, this is a second-order equation, of the form of the second-order expectational model, with $0 < a = c = 1/(1 + r) < 1$, $0 < b = \delta/(1 + r) < 1$ and $a + b < 1$. $x_i$ is a constant equal to $d_0$. 
The Fundamental Solution of the Second-Order Capital Market Model

The fundamental solution of model, using either the factorization method or the method of undetermined coefficients, gives us the stock price as a function of the past stock price and the constant part of dividends \(d_0\). This takes the form,

\[
p_t = \lambda p_{t-1} + \frac{\lambda}{\delta} \frac{1}{1 - \mu^{-1}F} d_0 = \lambda p_{t-1} + \frac{\lambda}{\delta} \sum_{s=0}^{\infty} \left( \frac{1}{\mu} \right)^s d_0
\]

The two roots \(\lambda\) and \(\mu\) lie on either side of unity, and satisfy \(\lambda + \mu = 1 + r\) and \(\lambda \mu = \delta\). \(\lambda\) is assumed to be the smaller root. Note that in the case where \(\delta = 0\), \(\lambda = 0\) and \(\mu = 1 + r\). With \(\delta = 0\) we are back to the first order case.

One can show that, since \(d_0\) is constant, the stock price evolves according to,

\[
p_t = \lambda p_{t-1} + (1 - \lambda) \frac{d_0}{r - \delta}
\]

The stock price converges to a steady state stock price, which is defined by,

\[
\bar{p} = \frac{d_0}{r - \delta}
\]

The steady state stock price is indeed higher than in the case where \(\delta = 0\), and the dividend policy of a positive \(\delta\) does indeed boost the price of the stock.